Band Surgeries between Knots and Links with Small Crossing Numbers

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During site-specific recombination, the topology of circular DNA can change, e.g. unknotted molecules can become knotted or linked. We model Xer site-specific recombinations as the mathematical operation of band surgeries. In this paper, we consider band surgeries on knots with 7 and fewer crossings and links with 8 and fewer crossings.

§1. Introduction

In DNA site-specific recombination, an enzyme attaches to a pair of DNA sites, and recombines the sites to different ends. During site-specific recombination, the topology of circular DNA can change, forming knots and links. We model Xer recombination as band surgery. By regarding Xer recombinations as band surgeries and applying mathematical results on band surgery, we confirm experimental results of Xer recombination acting on circular DNA. In particular, our motivation is the unlinking of DNA catenanes by Xer-*dif*-FtsK recombinations reported by Grainge et al.,[6)] and Xer recombination at the *psi*-site on DNA catenanes with $2k$ crossings which yields DNA knots with $2k+1$ crossings reported by Bath, Sherratt and Colloms.[1)] The main result of this paper is summarized in Table II of band surgeries between knots with 7 and fewer crossings and catenanes with 8 and fewer crossings. In §2, we relate band surgeries to site-specific recombinations. In §3, we give the table which characterize band surgeries. In §4, we give a table for band surgery between knots. For knots with 7 and fewer crossings, we use the classical notation as in the book by Rolfsen (see Appendix C in 14)). For a knot or link K, we denote by $K!$ the mirror image of K throughout this paper. We use several invariants of knots and links to construct Table II (see Theorems 3.4, 3.5, 3.7, 3.9, 3.11, and 3.13).

§2. Band surgery and site-specific recombination

Let L be a link in S^3 and $b : [0,1] \times [0,1] \to S^3$ be an embedding such that $b^{-1}(L) = [0,1] \times \{0,1\}$. Then we obtain a link L_b by replacing $b([0,1] \times \{0,1\})$ in L with $b(\{0,1\} \times [0,1])$ (see Fig. 1). We call this operation a *band surgery*. For simplicity we use the same symbol b to denote the image $b([0,1] \times [0,1])$. If L and L_b have the same orientations except for the band b, the band surgery is said to be *coherent*. A coherent band surgery results in a change of the number of link

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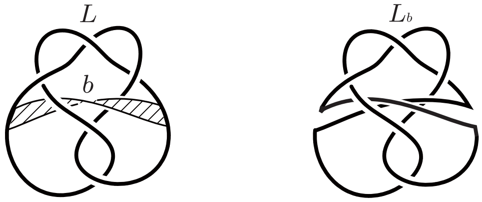


Fig. 1. A band surgery.

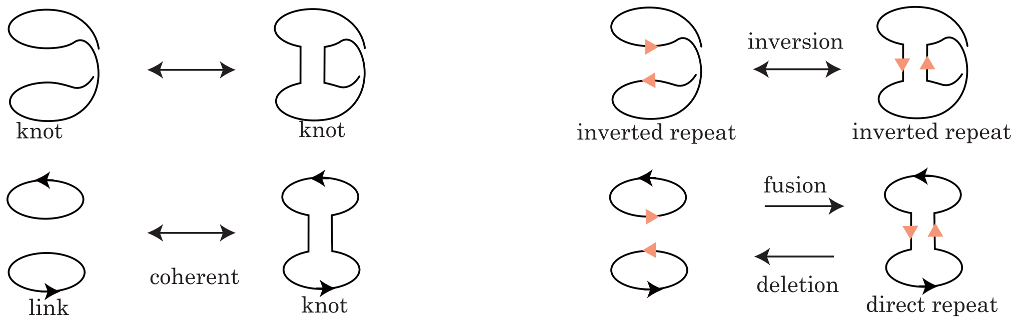


Fig. 2. Band surgeries and site-specific recombinations.

components. Indeed, any coherent band surgery on a knot yields a two component link. Moreover, for any band surgery from a knot to a two component link, we can take orientations so that the band surgery is coherent. On the other hand, any band surgery from a knot to a knot can not be coherent for any orientations. Band surgery between two knots is essentially equivalent to an $H(2)$-move, which is a local move for knots (see 9) for instance).

The DNA sequence of two recombination sites can be used to orient these sites. When the two sites are on the same component, their orientations on the circular DNA molecule can either agree or disagree. In the former case, we say that the two sites are directly repeated, while in the later case, we say that the two sites are inversely repeated. If two sites are directly repeated, we can take an orientation of the knot induced by the orientations of sites. By taking the orientation of two component link similarly after the recombination, we can regard a directly repeated Xer recombination as a coherent band surgery (see Fig. 2).

§3. Band surgery between knot and two component link

In this section, we deal with band surgeries between knots and two component links. As shown in the previous section, we can take orientations for a knot and a link so that the band surgery will be coherent. For a non-negative integer k, we denote by $T_{2k}, T'_{2k}, T_{2k}!, T'_{2k}!$ the oriented $(2,2k)$-torus links ($2k$-cats) as shown in Fig. 3. We call both of T_{2k} and $T_{2k}!$ parallel $2k$-cats, and call both of T'_{2k} and $T'_{2k}!$ anti-parallel $2k$-cats. Note that $T_0 = T'_0 = T_0! = T'_0!$ are trivial two component links, and $T_2 = T'_2!, T'_2 = T_2!$ are Hopf links.

By L_b, we denote the oriented link obtained from an oriented link L by the

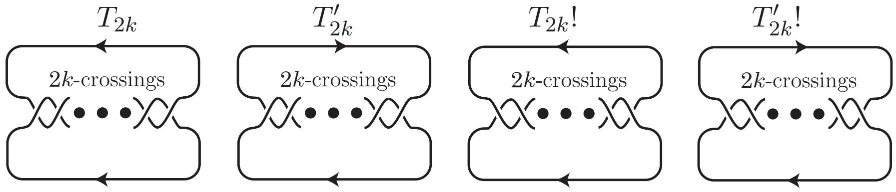


Fig. 3. $2k$-cats.

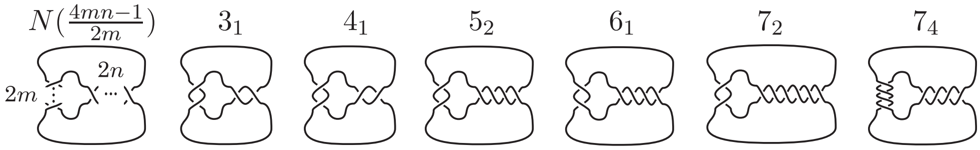


Fig. 4. Knots of type $N(\frac{4mn-1}{2m})$.

coherent band surgery along a band b, throughout this section. Since band surgeries are reversible, we may consider the band surgery as one from a knot to a catenane.

Theorem 3.1 (Scharlemann[15]). *Let L be a trivial knot. Then L_b is a trivial two component link if and only if the band b is trivial, i.e. there is a disk D such that $\partial D = L$ and $b \subset D$.*

Thompson[17] characterized the band surgery between a trivial knot and a Hopf link. Hirasawa and Shimokawa[7] generalized it for the case where L_b is a $(2, 2k)$-torus link.

Theorem 3.2 (Hirasawa-Shimokawa[7]). *Let L be a trivial knot. Then L_b is a $2k$-cat if and only if b is standard, i.e. there is a disk D such that $\partial D = L$, $b(\{\frac{1}{2}\} \times [0,1]) \subset D$, and b has $2k$ half twists with respect to D. (See the left hand side of Fig. 6.)*

Theorem 3.3 (Darcy-Ishihara-Medikonduri-Shimokawa[3]). *Suppose that L is a two bridge knot $N(\frac{4mn-1}{2m})$, where $N(\frac{p}{q})$ means the numerator of a $\frac{p}{q}$-tangle. Then L_b is isotopic to T'_{2k} only if $m = -k, n = -k, m+n+1 = -k$ or $m+n-1 = -k$, and L_b is isotopic to $T'_{2k}!$ only if $m = k, n = k, m+n+1 = k$ or $m+n-1 = k$.*

We also characterized the band b yielding an anti-parallel $2k$-cat from a knot of type $N(\frac{4mn-1}{2m})$.[3] We remark that a knot of type $N(\frac{4mn-1}{2m})$ is represented by vertical $2m$ full twists and horizontal $2n$ full twists as Fig. 4. Examples are the following:

$$3_1 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (1,1),\ 4_1 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (1,-1),$$
$$5_2 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (1,2),\ 6_1 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (-1,2),$$
$$7_2 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (1,-3), \text{ and } 7_4 = N(\tfrac{4mn-1}{2m}) \text{ for } (m,n) = (-2,-2).$$

Now we discuss the condition for existence of band surgeries between a knot and a two component link. Table I, which is taken from Table 2 in 9) and Table 1 in 8), lists several invariants of knots with 7 and fewer crossings and catenanes with 8 and fewer crossings; the signature which is denoted by $\sigma(\cdot)$, the special value of the Jones polynomial which is denoted by $V(\cdot\,;\omega)$, and the special value of the Q polynomial which is denoted by $\rho(\cdot)$. Murasugi[12] showed the following.

Theorem 3.4 (Murasugi[12]). $|\sigma(L) - \sigma(L_b)| \le 1$.

We denote the Alexandar polynomial of a knot K by $\Delta(K)$. The formula in

Theorem 3.5 is known as the Fox-Milnor condition for a *slice knot*, that bounds a properly embedded disk in B^4. Note that a knot which is obtained from a trivial two component link by a band surgery is a slice knot.

Theorem 3.5 (Fox-Milnor[5)]). *Suppose that a trivial two component link can be obtained from a knot K by a coherent band surgery. Then $\Delta_K(t) = \pm t^r f(t) f(t^{-1})$ for some integral polynomial $f(t)$.*

Together with Theorem 3.5, the following lemma implies that 6_3 and 7_7 have no band surgery yielding T_0.

Lemma 3.6.
(1) $\Delta_{6_3}(t) \neq \pm t^r f(t) f(t^{-1})$ for any integral polynomial $f(t)$.
(2) $\Delta_{7_7}(t) \neq \pm t^r f(t) f(t^{-1})$ for any integral polynomial $f(t)$.

In an equation, the symbol "$\doteq$" means equal up to multiplication by $\pm t^n$, throughout this paper.

Proof. (1) Using a proof by contradiction we assume $\Delta_{6_3}(t) \doteq 1 - 3t + 5t^2 - 3t^3 + t^4 = \varepsilon t^r f(t) f(t^{-1})$ for $\varepsilon = \pm 1$ and some integral polynomial $f(t)$. We may assume $r = 2$ and $f(t) = a + bt + ct^2$ for some integers a, b, and c. Then

$$1 - 3t + 5t^2 - 3t^3 + t^4 = ac\varepsilon + (ab + bc)\varepsilon t + (a^2 + b^2 + c^2)\varepsilon t^2 + (ab + bc)\varepsilon t^3 + ab\varepsilon t^4.$$

Hence $ac = \varepsilon$, $ab + bc = -3\varepsilon$. Since $ac = \varepsilon$, $a + c = 0$ or ± 2, and so $ab + bc = b(a + c) \neq -3\varepsilon$, a contradiction.

(2) Using a proof by contradiction we assume $\Delta_{7_7}(t) \doteq 1 - 5t + 9t^2 - 5t^3 + t^4 = \varepsilon t^r f(t) f(t^{-1})$ for $\varepsilon = \pm 1$ and some integral polynomial $f(t)$. We may assume $r = 2$ and $f(t) = a + bt + ct^2$ for some integers a, b, and c. Then

$$1 - 5t + 9t^2 - 5t^3 + t^4 = ac\varepsilon + (ab + bc)\varepsilon t + (a^2 + b^2 + c^2)\varepsilon t^2 + (ab + bc)\varepsilon t^3 + ab\varepsilon t^4.$$

Hence $ac = \varepsilon$, $ab + bc = -5\varepsilon$. Since $ac = \varepsilon$, $a + c = 0$ or ± 2, and so $ab + bc = b(a + c) \neq -5\varepsilon$, a contradiction.

□

Kawauchi showed that the knot 7_3 can not be obtained from a 6-cat by any band surgery, using Theorem 3.7 below. Theorem 3.7 follows from Theorem 3.5 and Theorem 1 in 13) .

Theorem 3.7 (Kawauchi). *Suppose a parallel $2k$-cat can be obtained from a knot K by a coherent band surgery. Then $\Delta_K(t) \equiv \pm t^r f(t) f(t^{-1}) \mod \frac{(1-t)(1-t^{2k})}{1-t^2}$.*

Together with Theorem 3.7, the following lemma implies that 7_3 has no band surgery yielding T_6 or $T_6!$.

Lemma 3.8 (Kawauchi). *$\Delta_{7_3}(t) \not\equiv \pm t^r f(t) f(t^{-1}) \mod \frac{(1-t)(1-t^6)}{1-t^2}$ for any integral polynomial $f(t)$.*

Proof. Using a proof of contradiction we assume $\Delta_{7_3}(t) \doteq 2 - 3t + 3t^2 - 3t^3 + 2t^4 = t^r f(t) f(t^{-1}) + (1-t)(1+t^2+t^4) g(t)$ for some integral polynomials $f(t), g(t)$. We may put $f(\omega) = a + b\omega$ for some integers a, b, because $\omega^2 = -1 + \omega$, where $\omega = e^{\pi i/3}$. Then $2 = |\Delta_{7_3}(\omega)| = |f(\omega) f(\omega^{-1})| = |f(\omega)|^2 = a^2 + b^2 + ab = \frac{(a+b)^2 + a^2 + b^2}{2}$. This equation cannot be satisfied for any integers a and b. □

Kawauchi[10] also showed that 7_6 can not be obtained from a 6-cat by any band surgery, by using Theorem 3.9 below.

Theorem 3.9 (Kawauchi[10]). *Suppose that an anti-parallel $2k$-cat can be obtained from a knot K by a coherent band surgery. Then $\Delta_K(t) \equiv \pm t^r f(t)f(t^{-1})$ (mod k).*

Lemma 3.10.
(1) $\Delta_{K_1}(t) \not\equiv \pm t^r f(t)f(t^{-1})$ (mod 2), for each knot $K_1 \in \{6_2, 6_3, 7_6, 7_7\}$.
(2) $\Delta_{K_2}(t) \not\equiv \pm t^r f(t)f(t^{-1})$ (mod 3), for each knot $K_2 \in \{7_6, 3_1\sharp 4_1\}$.
(3) $\Delta_{K_3}(t) \not\equiv \pm t^r f(t)f(t^{-1})$ (mod 4), for each knot $K_3 \in \{6_2, 6_3, 7_6, 7_7\}$.

Lemma 3.10 implies that K_i has no band surgery yielding T'_{2i+2} or $T'_{2i+2}!$ for each $i \in \{1, 2, 3\}$.

PROOF. We define integral polynomials $p_1(t), p_2(t), \cdots, p_5(t)$ as follows:

$$\begin{aligned}
p_1(t) &:= 1 - 3t + 3t^2 - 3t^3 + t^4 \doteq \Delta(6_2)\\
p_2(t) &:= 1 - 3t + 5t^2 - 3t^3 + t^4 \doteq \Delta(6_3)\\
p_3(t) &:= 1 - 5t + 7t^2 - 5t^3 + t^4 \doteq \Delta(7_6)\\
p_4(t) &:= 1 - 5t + 9t^2 - 5t^3 + t^4 \doteq \Delta(7_7)\\
p_5(t) &:= 1 - 4t - 5t^2 - 4t^3 + t^4 \doteq \Delta(3_1\sharp 4_1)
\end{aligned}$$

(1) We suppose that $p_i(t) \equiv \varepsilon t^r f(t)f(t^{-1})$ (mod 2) for $\varepsilon = \pm 1$ and some $i \in \{1, 2, 3, 4\}$. We may assume that $r = 2$ and $f(t) = a + bt + ct^2$ for some integers a, b, and c. Then

$$p_i(t) \equiv ac\varepsilon + (ab + bc)\varepsilon t + (a^2 + b^2 + c^2)\varepsilon t^2 + (ab + bc)\varepsilon t^3 + ab\varepsilon t^4 \pmod 2.$$

Hence $ac \equiv 1$ (mod 2) and $ab + bc \equiv 1$ (mod 2). Since $ac \equiv 1$ (mod 2), $a + c \equiv 0$ (mod 2), and so $ab + bc \equiv b(a + c) \equiv 0 \not\equiv 1$ (mod 2), a contradiction.

(2) We suppose that $p_i(t) \equiv \varepsilon t^r f(t)f(t^{-1})$ (mod 3) for $\varepsilon = \pm 1$ and some $i \in \{3, 5\}$. We may assume that $r = 2$ and $f(t) = a + bt + ct^2$ for some integers a, b, and c. Then

$$p_i(t) \equiv ac\varepsilon + (ab + bc)\varepsilon t + (a^2 + b^2 + c^2)\varepsilon t^2 + (ab + bc)\varepsilon t^3 + ab\varepsilon t^4 \pmod 3.$$

Hence $ac \equiv \varepsilon$ (mod 3), $ab + bc \equiv \begin{cases} \varepsilon & (i = 3) \\ -\varepsilon & (i = 5) \end{cases}$ (mod 3), and $a^2 + b^2 + c^2 \equiv \varepsilon$ (mod 3). Since $ac \equiv \pm 1$ (mod 3) and $b(a + c) = ab + bc \equiv \pm 1$ (mod 3), $a^2 \equiv b^2 \equiv c^2 \equiv 1$ (mod 3), and so $a^2 + b^2 + c^2 \equiv 0 \not\equiv \varepsilon$ (mod 3), a contradiction.

(3) We suppose that $p_i(t) \equiv \varepsilon t^r f(t)f(t^{-1})$ (mod 4) for $\varepsilon = \pm 1$ and some $i \in \{1, 2, 3, 4\}$. We may assume that $r = 2$ and $f(t) = a + bt + ct^2$ for some integers a, b, and c. Then

$$p_i(t) \equiv ac\varepsilon + (ab + bc)\varepsilon t + (a^2 + b^2 + c^2)\varepsilon t^2 + (ab + bc)\varepsilon t^3 + ab\varepsilon t^4 \pmod 4.$$

Hence $ac \equiv \varepsilon$ (mod 4), $ab + bc \equiv \begin{cases} \varepsilon & (i = 1, 2) \\ -\varepsilon & (i = 3, 4) \end{cases}$ (mod 4). Since $ac \equiv \pm 1$ (mod 4), $a + c \equiv 0$ (mod 2), and so $ab + bc \equiv b(a + c) \equiv 0$ or $2 \not\equiv \pm 1$ (mod 4), a contradiction. □

Table I. Invariants of knots and links: $\sigma(\cdot)$ means the signature, $V(\,\cdot\,;t)$ means the Jones polynomial and $\omega = e^{\pi i/3}$, and $\rho(\cdot) = Q(\,\cdot\,;(\sqrt{5}-1)/2)$ where $Q(\,\cdot\,;t)$ means the Q polynomial.

	$\sigma(K)$	$V(K;\omega)$	$\rho(K)$
0_1	0	1	1
3_1	2	$-i\sqrt{3}$	-1
4_1	0	-1	$-\sqrt{5}$
5_1	4	-1	$\sqrt{5}$
5_2	2	-1	-1
6_1	0	$i\sqrt{3}$	1
6_2	2	1	1
6_3	0	1	-1
$3_1\sharp 3_1$	4	-3	1
$3_1\sharp 3_1!$	0	3	1
7_1	6	-1	-1
7_2	2	1	1
7_3	-4	1	-1
7_4	-2	$-i\sqrt{3}$	$\sqrt{5}$
7_5	4	-1	-1
7_6	2	-1	1
7_7	0	$-i\sqrt{3}$	1
$3_1\sharp 4_1$	2	$i\sqrt{3}$	$\sqrt{5}$

L	$\sigma(L)$	$V(L;\omega)$	$\rho(L)$
T_0	0	$-\sqrt{3}$	$\sqrt{5}$
T_2	1	i	-1
T_2'	-1	$-i$	-1
T_4	3	$-i$	1
T_4'	-1	$-i$	1
$T_4!$	-3	i	1
$T_4'!$	1	i	1
T_6	5	$\sqrt{3}$	1
T_6'	-1	$-\sqrt{3}$	1
$T_6!$	-5	$\sqrt{3}$	1
$T_6'!$	1	$-\sqrt{3}$	1
T_8	7	i	-1
T_8'	-1	i	-1
$T_8!$	-7	$-i$	-1
$T_8'!$	1	$-i$	-1

We denote the Jones polynomial of a link L by $V(L;t)$, and put $\omega = e^{\pi i/3}$. Kanenobu[8)] proved Theorem 3.11 below which can be used to show that 7_3 and 7_6 (resp. 9_{15} and 9_{17}) cannot be obtained from a 6-cat (resp. 8-cat) by any band surgery.

Theorem 3.11 (Kanenobu[8)]). $V(L;\omega)/V(L_b;\omega) \in \{\pm i, -\sqrt{3}^{\pm 1}\}$.

Lemma 3.12. *(1)* $V(K_1;\omega)/V(L_1;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for each knot* $K_1 \in \{3_1, 7_4, 7_7\}$ *and each link* $L_1 \in \{T_2', T_4, T_4', T_8!, T_8'!\}$.

(2) $V(K_2;\omega)/V(L_2;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for each knot* $K_2 \in \{4_1, 5_1, 5_2, 7_1, 7_5, 7_6\}$ *and link* $L_2 \in \{T_0, T_6', T_6'!\}$.

(3) $V(K_3;\omega)/V(L_3;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for each knot* $K_3 \in \{6_1, 3_1\sharp 4_1\}$ *and link* $L_3 \in \{T_2, T_4!, T_4'!, T_8, T_8'\}$.

(4) $V(K;\omega)/V(L;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for each knot* $K_4 \in \{6_2, 6_3, 7_2, 7_3\}$ *and link* $L_4 \in \{T_6, T_6!\}$.

(5) $V(K_5;\omega)/V(L_5;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for a knot* $K_5 = 3_1\sharp 3_1$ *and each link* $L_5 \in \{T_0, T_2, T_2', T_4, T_4', T_4!, T_4'!, T_6', T_6'!, T_8, T_8', T_8!, T_8'!\}$.

(6) $V(3_1\sharp 3_1!;\omega)/V(L_6;\omega) \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$, *for a knot* $K_6 = 3_1\sharp 3_1!$ *and each link* $L_6 \in \{T_2, T_2', T_4, T_4', T_4!, T_4'!, T_6, T_6!, T_8, T_8', T_8!, T_8'!\}$.

Lemma 3.12 implies that there is no band surgery between a knot K_i and a link L_i for each $i \in \{1,2,3,4,5,6\}$.

Proof. From Table I, $V(K_i;\omega)/V(L_i;\omega) = \sqrt{3}^{\pm 1}$ or $\pm 3i \notin \{\pm i, -\sqrt{3}^{\pm 1}\}$ for each $i \in \{1,2,3,4,5,6\}$. $\square$

Kanenobu[8)] proved Theorem 3.13 below which can be used to show that 9_{31} can not be obtained from a 8-cat by any band surgery.

Theorem 3.13 (Kanenobu[8]). $\rho(L)/\rho(L_b) \in \{\pm 1, \sqrt{5}^{\pm 1}\}$.

Lemma 3.14.

(1) $\rho(K_1)/\rho(L_1) \notin \{\pm 1, \sqrt{5}^{\pm 1}\}$, *for a knot* $K_1 = 4_1$ *and each link* $L_1 \in \{T_4, T_4', T_4!, T_4'!, T_6, T_6', T_6!, T_6'!\}$.

(2) $\rho(K_2)/\rho(L_2) \notin \{\pm 1, \sqrt{5}^{\pm 1}\}$, *for each knot* $K_2 \in \{5_1, 7_4, 3_1\sharp 4_1\}$ *and each link* $L_2 \in \{T_2, T_2', T_8, T_8', T_8!, T_8'!\}$.

Lemma 3.14 implies that there is no band surgery between a knot K_i and a link L_i for each $i = \{1, 2\}$.

Proof. From Table I, $\rho(K_i)/\rho(L_i) = -\sqrt{5} \notin \{\pm 1, \sqrt{5}^{\pm 1}\}$ for each $i = \{1, 2\}$. □

From Theorem 3.4, Lemmas 3.6, 3.8, 3.10, 3.12, and 3.14, we can show the non-existence of band surgeries which is denoted by the symbol $\times$ in Table II. From Figs. 6 (the left hand side), 7, 8, 9, and 10, we can show the existence of band surgeries denoted by the symbols $\ominus$ and $\circledcirc$ in Table II. We note that there is a band surgery on a knot K which yields a 2-cat if K is unknotting number one. Because K has a crossing on which the crossing change produces an unknot, then we obtain a 2-cat by band surgery on that crossing as shown in Fig. 5.

Table II. Band surgeries between knots and catenanes: The symbol $\circledcirc^{k)}$ implies there are band surgeries, and they are completely characterized in k). The symbol $\ominus$ implies there are band surgeries, but they are not characterized yet. The symbol $\times$ implies there is no band surgery.

	T_0	T_2	T_2'	T_4	T_4'	$T_4!$	$T_4'!$	T_6	T_6'	$T_6!$	$T_6'!$	T_8	T_8'	$T_8!$	$T_8'!$
0_1	$\circledcirc^{15)}$	$\circledcirc^{17)}$	$\circledcirc^{17)}$	$\times$	$\circledcirc^{7)}$	$\times$	$\circledcirc^{7)}$	$\times$	$\circledcirc^{7)}$	$\times$	$\circledcirc^{7)}$	$\times$	$\circledcirc^{7)}$	$\times$	$\circledcirc^{7)}$
3_1	$\times$	$\circledcirc^{3)}$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$
4_1	$\times$	$\circledcirc^{3)}$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
5_1	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
5_2	$\times$	$\circledcirc^{3)}$	$\times$		$\times$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\circledcirc^{3)}$
6_1	$\circledcirc^{3)}$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
6_2	$\times$	$\ominus$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$
6_3	$\times$	$\ominus$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$		$\times$	$\times$	$\times$	$\times$
$3_1\sharp 3_1$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$3_1\sharp 3_1!$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$		$\times$	$\times$	$\times$	$\times$
7_1	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$
7_2	$\times$	$\circledcirc^{3)}$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$
7_3	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
7_4	$\times$	$\times$	$\times$	$\times$	$\circledcirc^{3)}$		$\times$	$\times$	$\circledcirc^{3)}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
7_5	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
7_6	$\times$	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
7_7	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$		$\times$	$\times$	$\times$	$\times$
$3_1\sharp 4_1$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

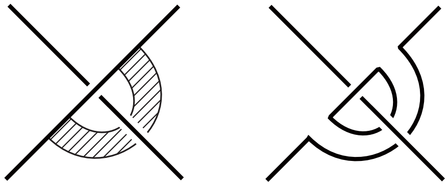
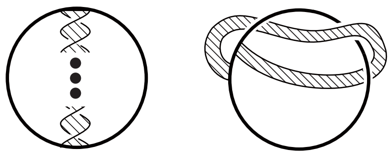
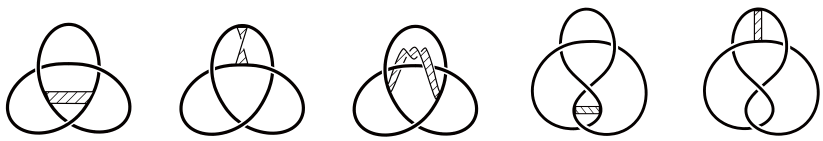


Fig. 5. A band attaching at a crossing.



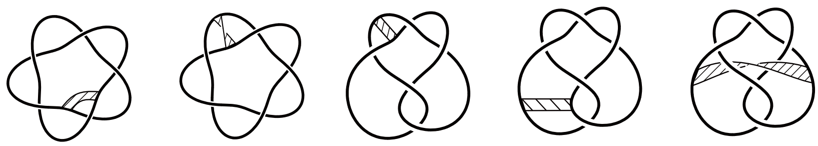
$0_1 \to T_p$ $0_1 \to 5_1$

Fig. 6. Bands attaching on unknots.



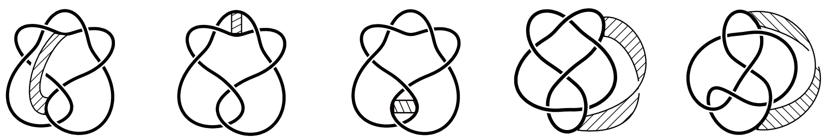
$3_1 \to T_2$ $3_1 \to T_4$ $3_1 \to T_6'!$ $4_1 \to T_2$ $4_1 \to T_2'$

Fig. 7. Bands attaching on 3_1 and 4_1.

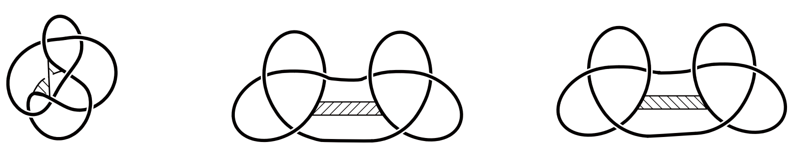


$5_1 \to T_4$ $5_1 \to T_6$ $5_2 \to T_2$ $5_2 \to T_4'!$ $5_2 \to T_8'!$

Fig. 8. Bands attaching on 5 crossing knots.



$6_1 \to T_0$ $6_1 \to T_2'$ $6_1 \to T_4'!$ $6_2 \to T_2$ $6_3 \to T_2$



$6_3 \to T_2'$ $3_1\#3_1 \to T_6$ $3_1\#3_1! \to T_0$

Fig. 9. Bands attaching on 6 crossing knots.

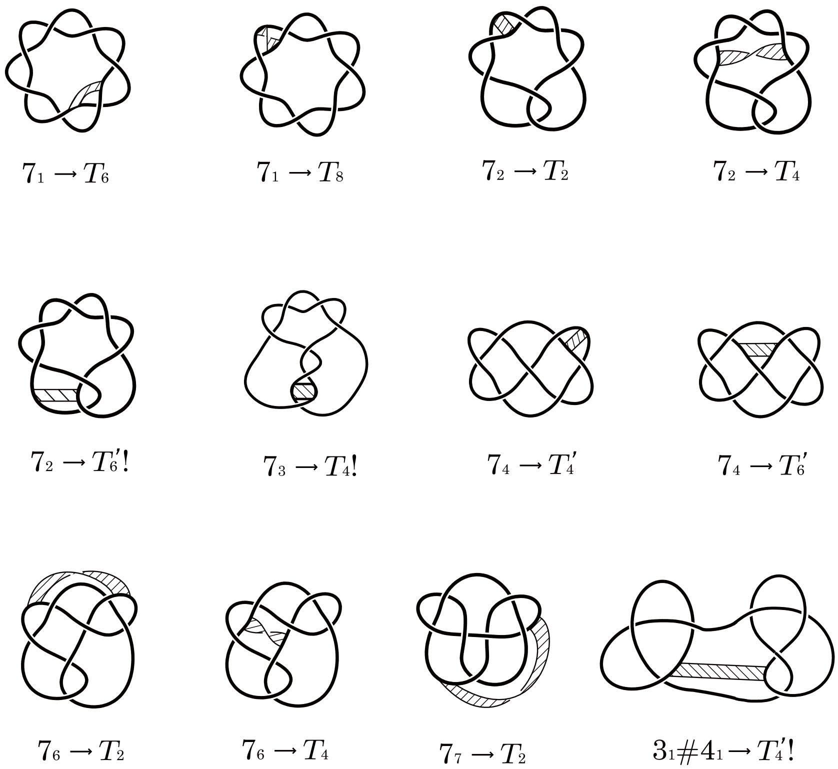


Fig. 10. Bands attaching on 7 crossing knots.

§4. Band surgery between two knots

Any band surgery between two knots is essentially equivalent to an $H(2)$-move. The minimum number of $H(2)$-moves needed to transform one knot into another knot is called the $H(2)$-*Gordian distance.* Kanenobu[9)] gave a table of $H(2)$-Gordian distances for knots with 7 and fewer crossings. Table III is copied from 9) focussing only whether or not there exists each band surgery. In this section, we characterize band surgeries from the unknot to $(2, p)$-torus knot.

Since any knot in S^3 having a Dehn surgery which yields a lens space $L(p, 1)$ is determined completely by Theorem 1.1 in 11) and Theorem 9 in 16), we obtain the following theorem.

Theorem 4.1. *Let L be a trivial knot. Then L_b is $(2, p)$-torus knot if and only if b is standard or $p = \pm 5$ and $L \cup b$ is isotopic to the right hand side of Fig. 6 or its mirror image.*

Proof. (Proof of the "if" part.) We can see that L_b is isotopic to a $(2, -5)$-torus knot (5_1) for the band b as shown in Fig. 6 on the right.

(Proof of the "only if" part.) Let B be a regular neighborhood of the band b. Then both of two pairs $(B, B \cap L)$ and $(B, B \cap L_b)$ are rational tangles. By moving b to a vertical twisted band, $(B, B \cap L)$ and $(B, B \cap L_b)$ can be considered as the 0-tangle and the $\frac{1}{w}$-tangle respectively, and so we can regard the band surgery from L

Table III. Band surgeries between knots.

	3_1	4_1	5_1	5_2	6_1	6_2	6_3	$3_1\sharp 3_1$	$3_1\sharp 3_1!$	7_1	7_2	7_3	7_4	7_5	7_6	7_7	$3_1\sharp 4_1$
0_1	$\circledcirc$	$\times$	$\circledcirc$	$\ominus$	$\ominus$	$\ominus$	$\times$	$\times$	$\times$	$\circledcirc$	$\ominus$	$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\ominus$
3_1	$\ominus$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$
$3_1!$	$\times$	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$	$\ominus$	$\times$	$\ominus$	$\times$	$\ominus$	$\times$
4_1	$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$	$\ominus$
5_1	$\times$	$\times$	$\ominus$	$\times$		$\ominus$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$
$5_1!$	$\times$	$\times$		$\times$	$\ominus$		$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$		$\times$	$\times$
5_2	$\times$	$\ominus$	$\times$	$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$		$\times$	$\ominus$	$\times$	$\ominus$	$\times$
$5_2!$	$\ominus$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$				$\times$	$\ominus$	$\ominus$	$\times$	$\times$
6_1	$\times$	$\times$		$\ominus$	$\ominus$		$\times$	$\times$	$\ominus$	$\times$	$\ominus$	$\ominus$		$\times$	$\times$	$\times$	$\times$
$6_1!$	$\times$	$\times$	$\ominus$	$\times$		$\times$	$\times$	$\times$	$\ominus$		$\times$	$\ominus$		$\times$	$\ominus$	$\times$	$\times$
6_2	$\ominus$	$\times$	$\ominus$	$\times$		$\ominus$	$\ominus$	$\times$	$\times$				$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\times$
$6_2!$	$\times$	$\times$		$\ominus$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$		$\times$	$\ominus$	$\times$	$\times$	
6_3	$\times$	$\ominus$	$\times$	$\ominus$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$			$\times$	$\times$		$\ominus$	$\ominus$	$\times$
$3_1\sharp 3_1$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\ominus$	$\ominus$
$3_1!\sharp 3_1!$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$		$\times$	$\times$		
$3_1\sharp 3_1!$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\ominus$
7_1	$\ominus$	$\ominus$	$\times$	$\times$	$\times$			$\times$	$\times$	$\ominus$			$\times$			$\times$	$\times$
$7_1!$	$\times$	$\ominus$	$\times$			$\times$		$\times$	$\times$	$\times$	$\times$		$\times$		$\times$		$\times$
7_2	$\ominus$	$\times$		$\times$	$\ominus$			$\times$	$\times$		$\ominus$		$\times$				$\times$
$7_2!$	$\times$	$\times$	$\ominus$		$\times$	$\times$		$\times$	$\times$	$\times$	$\times$		$\times$		$\times$	$\times$	$\ominus$
7_3	$\ominus$	$\times$	$\times$		$\ominus$		$\times$	$\times$	$\times$			$\ominus$		$\times$		$\times$	$\times$
$7_3!$	$\ominus$	$\times$	$\times$		$\ominus$		$\times$	$\times$	$\times$				$\times$	$\times$		$\times$	$\times$
7_4	$\times$	$\ominus$	$\times$	$\times$		$\ominus$	$\times$	$\ominus$		$\times$		$\times$	$\ominus$	$\times$		$\ominus$	$\times$
$7_4!$	$\times$	$\ominus$	$\times$	$\times$		$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		
7_5	$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\ominus$		$\times$	$\times$			$\times$	$\times$	$\ominus$			$\times$
$7_5!$	$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\ominus$		$\times$	$\times$			$\times$	$\times$				$\times$
7_6	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$						$\ominus$	$\times$	$\times$
$7_6!$	$\times$	$\times$		$\ominus$	$\ominus$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$		$\times$			$\ominus$	$\ominus$
7_7	$\times$	$\times$	$\times$	$\ominus$	$\times$	$\ominus$	$\ominus$	$\ominus$	$\times$	$\times$		$\times$	$\ominus$		$\times$	$\ominus$	
$7_7!$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$		$\times$		$\times$	$\times$			$\ominus$	$\times$	
$3_1\sharp 4_1$	$\times$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\ominus$	$\ominus$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		$\ominus$
$3_1!\sharp 4_1$	$\times$	$\ominus$	$\times$	$\times$	$\times$		$\times$		$\ominus$	$\times$	$\ominus$	$\times$		$\times$	$\ominus$		$\times$

to L_b as a rational tangle surgery from $N(U+0) = L$ to $N(U+\frac{1}{w}) = L_b$, where U is a 2-string tangle, $+$ means the sum of two tangles, $N(\cdot)$ means the numerator, and w is a integer. Since L is a trivial knot, the 2-fold branched covering of S^3 branched over L is homeomorphic to S^3. Let K be the core knot of the solid torus which is obtained from the 0-tangle on $N(U+0)$ by taking the 2-fold branched covering. Then the 2-fold branched covering of S^3 branched over L_b, which is homeomorphic to $L(p,1)$, is obtained from S^3 by Dehn surgery on K. By Theorem 1.1 in 11) and Theorem 9 in 16), K is a trivial knot or $p = \pm 5$ and K is a trefoil knot. Since 2-fold branched covering of the tangle U is homeomorphic to the exterior of K, by Theorem 8 in 4), U is a rational tangle or $p = \pm 5$ and U is homeomorphic to a sum of two rational tangles. In the case where U is a rational tangle, we may assume $w = -p$, and so $N(U) = N(U+0) = N(\infty)$ (unknot) and $N(U+(-\frac{1}{p})) = N(\infty+(-\frac{1}{p}))$ ($(2,p)$-torus knot). By the first condition $N(U) = N(\infty)$ we obtain $U = (\frac{1}{k})$ for some integer k.

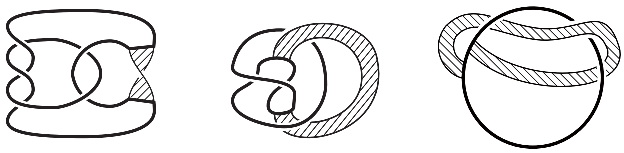


Fig. 11.

By putting $U = (\frac{1}{k})$ in the second condition $N(U+(-\frac{1}{p})) = N(\infty+(-\frac{1}{p}))$, we obtain $k = 0$ because the left is $(2, p-k)$-torus knot or link. Then U is the ∞-tangle. It means that b is standard. In the case where $p = \pm 5$ and U is homeomorphic to a sum of two rational tangle, we may assume that $p = -5$ and $w = 1$. Applying Theorem 3 in 2) we obtain the solutions $U = (\frac{1}{3}) + (-\frac{1}{2})$ and $U = (-\frac{1}{2}) + (\frac{1}{3})$, otherwise U is a rational tangle. Then, Fig. 11 illustrates that $L \cup b$ is isotopic to the left hand side of Fig. 6. $\square$

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